1. Define $f : \mathbb{C} \to \mathbb{C}$ by $f(x+iy) = (x + e^x \cos y) + i(e^x \sin y - y)$ for all $x, y \in \mathbb{R}$.

(a) (10 pts) Determine the set of all points in $\mathbb{C}$ at which f is complex differentiable.

Let $u(x,y) = x + e^x \cos y$ and $v(x,y) = e^x \sin y - y$ so that

$f(x+iy) = u(x,y) + i v(x,y)$

f is $\mathbb{C}$-differentiable at any point $x+iy \in \mathbb{C}$ if and only if both $u(x,y)$ and $v(x,y)$ are $\mathbb{R}$-differentiable at (x,y) and they satisfy the Cauchy - Riemann equations.

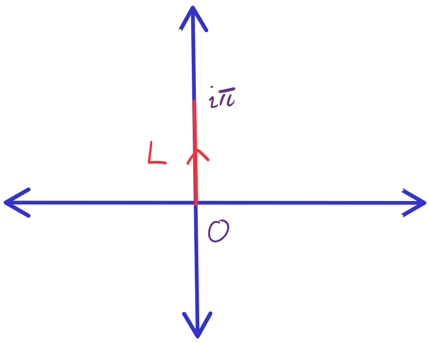
$u_x = v_y \qquad v_x = -u_y$

Notice $u_x = 1 + e^x \cos y$ and $v_y = e^x \cos y - 1$

$u_x(x,y) \neq v_y(x,y)$ for any (x,y).

Therefore the set $\{z \in \mathbb{C} : f \text{ is } \mathbb{C}\text{-differentiable at } z\}$ is the empty set.

(b) (10 pts) Compute the line integral $\int_L f(z)\,dz$ where L is the line segment from 0 to $i\pi$ in $\mathbb{C}$.



L can be parametrized by

$z = (1-t)(0) + t(i\pi)$ for $0 \le t \le 1$

$\Rightarrow x + iy = i t\pi$

$\Rightarrow x = 0 \qquad y = t\pi \qquad 0 \le t \le 1.$

$dz = d(x+iy) = d(0 + i t\pi) = i\pi\, dt.$

$$\int_L f(z)\,dz = \int_{t=0}^{1} f(0 + it\pi)\, i\pi\, dt = \int_{t=0}^{1} \left[\cos(\pi t) + i\sin(\pi t) - i\pi t\right] i\pi\, dt$$

$$= \int_{t=0}^{1} i\pi \cos(\pi t) - \pi \sin(\pi t) + \pi^2 t \; dt = \left[i\sin(\pi t) + \cos(\pi t) + \frac{\pi^2 t^2}{2} \right]_{t=0}^{1}$$

$$= -1 + \frac{\pi^2}{2} - 1 = \boxed{\frac{\pi^2}{2} - 2}$$

2. (15 pts) Let C denote the circle $|z| = 3$ oriented counterclockwise. Compute the integral

$$\int_C \frac{\sin(z^2)}{z^3(z-1)}\,dz.$$

The function $f(z) = \dfrac{\sin(z^2)}{z^3(z-1)}$ has isolated singularities at $z=0$ and $z=1$, and it is complex analytic at any other points. Since both $z=0$ and $z=1$ lie inside of C, the residue theorem tells that

$$\int_C f(z)\,dz = 2\pi i\,(\,\mathrm{Res}(f,0) + \mathrm{Res}(f,1)\,) \quad ☆$$

It remains to compute the above residues:

At $z=0$: f has a pole of order 1 since $\lim_{z\to 0} f(z)$ does not exist,

$$\lim_{z\to 0} z f(z) = \lim_{z\to 0} \frac{\sin(z^2)}{z^2}\,\frac{1}{z-1} = (1)\,(-1) = -1, \text{ and}$$

$$\lim_{z\to 0} z^2 f(z) = \lim_{z\to 0} z\,(z f(z)) = 0\,(-1) = 0.$$

So write $f(z) = \dfrac{g(z)}{z}$ for some analytic $g(z)$ with $g(0) \neq 0$ $\quad \mathrm{Res}(f,0) = g(0) = \lim_{z\to 0} z f(z) = -1$.

At $z=1$, f has a simple pole and $\mathrm{Res}(f,1) = \dfrac{\sin(1^2)}{(1)^3} = \sin(1)$

substituting these in ☆

$$\boxed{\int_C f(z)\,dz = 2\pi i\,(\,-1 + \sin(1)\,)}$$

3. (15 pts) Use Rouché's theorem to to determine the number of zeros, counting with multiplicities, of the polynomial $z^4 - 2z^3 + 9z^2 + z - 1$ inside the disk $|z| < 2$.

Let $f(z) = 9z^2$ and $g(z) = z^4 - 2z^3 + z - 1$ so that $f(z) + g(z)$ is the polynomial in the problem.

For any point on the circle $|z| = 2$

$$|g(z)| = |z^4 - 2z^3 + z - 1| \leq |z|^4 + 2|z|^3 + |z| + 1 \leq 16 + 16 + 2 + 1 = 35$$

$$|f(z)| = |9z^2| = 9|z|^2 = 9(4) = 36$$

Hence $|f(z)| > |g(z)|$ for all z such that $|z| = 2$

By Rouché's thm, $f(z)$ and $f(z) + g(z)$ have the same number of zeros in the disk $|z| < 2$. Clearly $f(z) = |z|^4$ has exactly 4 zeros, counted with multiplicity, located at the origin. Therefore $f(z) + g(z)$ has exactly 4 zeros in the disk $|z| < 2$.

4. (15 pts) Suppose f and g are entire functions and D is a compact subset of $\mathbb{C}$. Show that $|f(z)| + |g(z)|$, for $z \in D$ takes its maximum value on the boundary of D. (**Suggestion:** Consider $f(z)e^{i\alpha} + g(z)e^{i\beta}$ for appropriate α and β.)

If both $f(z)$ and $g(z)$ are constant functions. then the statement is trivially true. Suppose one of f and g is not constant. Consider the real valued function $\phi : D \to \mathbb{R}$ defined by $\phi(z) = |f(z)| + |g(z)|$. Since ϕ is a continuous function and its domain D is a compact set, ϕ attains its maximum at some point $z_0 \in D$, so that

$$|f(z)| + |g(z)| \leq |f(z_0)| + |g(z_0)| \quad \text{for all } z \in D. \quad \star$$

We'll show z_0 must be an element of the boundary of D. Let $\alpha, \beta \in [0, 2\pi)$ so that

$$e^{i\alpha} f(z_0) = |f(z)| \quad \text{and} \quad e^{i\beta} g(z_0) = |g(z)|$$

Consider the function $h(z) = e^{i\alpha} f(z) + e^{i\beta} g(z)$. Since both $f(z)$ and $g(z)$ are entire functions and $e^{i\alpha}$ and $e^{i\beta}$ are fixed complex numbers $h(z)$ is also entire.

We also have $|h(z)| \leq |e^{i\alpha} f(z) + e^{i\beta} g(z)| \leq |f(z)| + |g(z)|$
Combining this with $\star$, we see that

$$|h(z)| \leq |f(z_0)| + |g(z_0)| = e^{i\alpha} f(z_0) + e^{i\beta} g(z_0) = h(z_0), \text{ for all } z \in D.$$

Hence the entire function $h(z)$ attains its maximum modulus in D, at z_0. Since $h(z)$ is not a constant function, the maximum modulus theorem implies z_0 is a boundary point of D.

5. (a) (10 pts) Let C_R denote the half circle $\{z : |z| = R,\ \mathrm{Im}(z) \geq 0\}$. Prove that

$$\lim_{R\to\infty} \int_{C_R} \frac{e^{iz}}{z^2+1}\,dz = 0$$

For $z \in C_R$, we have $|z| = R$ and $z = R\cos\theta + iR\sin\theta$ for some $0 \leq \theta \leq \pi$. Then

$$\left| \frac{e^{iz}}{z^2+1} \right| = \frac{|e^{iz}|}{|z^2+1|} \leq \frac{|e^{iz}|}{|z|^2-1} = \frac{\overbrace{|e^{iR\cos\theta}|}^{=1}\ \overbrace{e^{-R\sin\theta}}^{\leq 1}}{R^2-1}$$

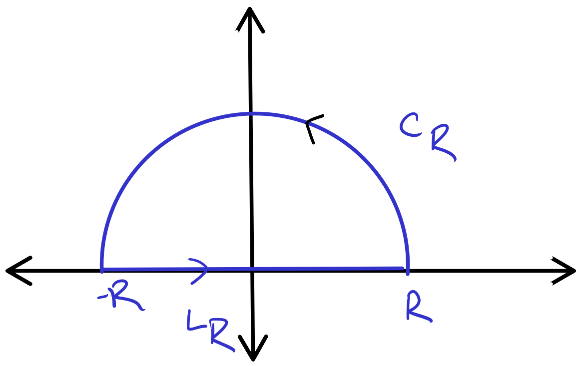
$$\leq \frac{1}{R^2-1}$$

$$\Rightarrow \left| \int_{C_R} \frac{e^{iz}}{z^2+1}\,dz \right| \leq \frac{1}{R^2-1}\, 2\pi R \to 0 \text{ as } R \to \infty.$$

(b) (15 pts) Using complex residues and part (a), compute the (real valued) integral

$$\int_{-\infty}^{\infty} \frac{\cos x}{x^2+1}\,dx.$$

Define $f(z) = \dfrac{e^{iz}}{z^2+1}$ which has simple poles at $z = \mp i$ and holomorphic on $\mathbb{C}\setminus\{\mp i\}$. Let $\gamma_R = C_R \cup L_R$



C_R: $z = Re^{i\theta}$ $\quad 0 \leq \theta \leq \pi$

L_R: $z = x$ $\quad -R \leq x \leq R$

We have

$$\int_{\gamma_R} f(z)\,dz = \int_{C_R} f(z)\,dz + \int_{L_R} f(z)\,dz \qquad (*)$$

$$\int_{\gamma_R} f(z)\,dz = 2\pi i\,(\mathrm{Res}\, f(z), i)$$ (since $-i$ does not lie inside γ_R) for all $R > 1$

$$\mathrm{Res}(f(z), i) = \mathrm{Res}\left(\frac{e^{iz}}{z^2+1}, i\right) = \left[\frac{e^{iz}}{z+i}\right]_{z=i} = \frac{e^{-1}}{2i} \qquad (**)$$

$$\int_{L_R} f(z)\,dz = \int_{x=-R}^{R} \frac{e^{ix}}{x^2+1}\,dx = \int_{x=-R}^{R} \frac{\cos x}{x^2+1}\,dx + i\int_{x=-R}^{R} \frac{\sin x}{x^2+1}\,dx \qquad (***)$$

Letting $R \to \infty$ in both sides of $(*)$ and combining the result of part(a) with $(**)$ and $(***)$, we get

$$2\pi i\left(\frac{e^{-1}}{2i}\right) = 0 + \int_{-\infty}^{\infty} \frac{\cos x}{x^2+1}\,dx + i\int_{-\infty}^{\infty} \frac{\sin x}{x^2+1}\,dx \qquad (4)$$

Notice $\frac{|\cos x|}{x^2+1} \le \frac{1}{x^2+1}$ and $\frac{|\sin x|}{x^2+1} \le \frac{1}{x^2+1}$

and $\int_{-\infty}^{\infty} \frac{1}{x^2+1}\,dx$ converges, so both $\int_{-\infty}^{\infty} \frac{\cos x}{x^2+1}\,dx$ and $\int_{-\infty}^{\infty} \frac{\sin x}{x^2+1}\,dx$

are absolutely convergent. Hence by (4), we see

$$\boxed{\int_{-\infty}^{\infty} \frac{\cos x}{x^2+1}\,dx = \pi e^{-1}}$$

Remark:

As a byproduct we also get

$$\int_{-\infty}^{\infty} \frac{\sin x}{x^2+1}\,dx = 0$$ which is not so difficult to see in the first place because it is the integral of an odd function in a symmetric interval